Quantum quenches in the Dicke model: Statistics of the work done and of other observables

Francis N. C. Paraan* and Alessandro Silva†
The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34151 Trieste, Italy
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We study the statistics of the work done in a zero temperature quench of the coupling constant in the Dicke model describing the interaction between an ensemble of two level systems and a single bosonic mode. When either the final or the initial coupling constants approach the critical coupling λ_c that separates the normal and superradiant phases of the system, the probability distribution of the work done displays singular behavior. The average work tends to diverge as the initial coupling parameter is brought closer to the critical value λ_c. In contrast, for quenches ending close to criticality, the distribution of work has finite moments but displays a sequence of edge singularities. This contrasting behavior is related to the difference between the processes of compression and expansion of a particle subject to a sudden change in its confining potential. We confirm this by studying in detail the time-dependent statistics of other observables, such as the quadratures of the photons and the total occupation of the bosonic modes.

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I. INTRODUCTION

The study of nonequilibrium phenomena in interacting quantum systems is one of the most challenging problems of modern statistical physics. The main reason is that several of the conceptual tools developed to describe physical systems in equilibrium (e.g., the partition function, mean field theory, and the renormalization group) are not readily generalized to nonequilibrium conditions. In order to make some progress, in our understanding of nonequilibrium behavior, it is important to identify simple paradigms of nonequilibrium processes that may be studied both theoretically and experimentally. Recently, some progress in this direction has been made through the realization of nonequilibrium experiments with cold atomic gases loaded in optical lattices [1–3]. To a good degree of accuracy, these systems are well-described by simple many-body models such as, for example, the Bose-Hubbard model [4].

The simplest nonequilibrium process among those presently under study is the *quantum quench*: an abrupt change in time of one of the system parameters from an initial value λ to a final one λ'. In a closed system, this process corresponds to the preparation of the system in the ground state $|0_\lambda\rangle$ of an initial Hamiltonian $H[\lambda]$, which is then allowed to evolve in time according to a final Hamiltonian $H[\lambda']$. This process is particularly interesting if some qualitative changes in the state of the system occur between λ and λ'. This was the case in Ref. [2], where a gas of bosonic atoms was taken abruptly across a quantum critical point from the superfluid to the Mott insulating region of the phase diagram. The observation of intriguing many-body collapse and revival cycles of the two phases in the momentum distribution function signaled the high degree of many-body coherence in the dynamics of these systems [1].

Theoretically, processes assimilable to quantum quenches have already been studied a few decades ago in a series of seminal papers [5]. More recently, however, the experiments discussed above have inspired an impressive and rapidly growing activity on this subject [6–10,17]. Apart from the concrete possibility of testing theoretical results with experiments, the main motivation behind this interest has been the shift in focus toward a broad class of fundamental issues. More specifically, a number of recent studies addressed the extension of the concept of universality to the out of equilibrium behavior of quantum critical systems subject to either quenches at or close to criticality [6] or to linear sweeps of the control parameter across the quantum critical point [11]. Similarly, a great deal of activity is devoted to the search for dynamical manifestations of quantum integrability and to the study of the relation between nonintegrability and thermalization [7].

Looking for a simple and fundamental way to characterize quantum quenches from the point of view of nonequilibrium physics, it was recently observed that a quantum quench may be considered in the context of basic statistical mechanics as a simple thermodynamic transformation [8–10]. It is thus quite natural to characterize quantum quenches using standard thermodynamic variables: the work W done on the system [8], the entropy S produced [9], and the heat Q generated [10]. Focusing on the work done, the abruptness of the change in the system parameters in a quantum quench implies that measurements of the work done in different realizations of the same protocol, defined as the difference of internal energies before and after the quench, will necessarily display fluctuations. This is also the case in classical nonequilibrium systems [12] and is in evident contrast with quasistatic/adiabatic processes where the system remains at all times in the ground state and the work done reduces to the difference in ground state energies before and after the quench. Therefore, for a complete characterization of a quench, it is not enough to specify the average work done on the system, but it is necessary to specify the full probability distribution of the work $P(W)$, which has to satisfy a number of constraints, such as the Jarzynski equalities and the Tasaki-Crooks fluctuation theorem [12]. For an abrupt quantum quench, $P(W)$ takes a particularly simple form

*Present address: State University of New York at Stony Brook, New York 11794, USA.
†asilva@ictp.it

$$P(W) = \sum_n |\langle n_{\lambda'}|0_\lambda\rangle|^2 \delta\{W - [E_n(\lambda') - E_0(\lambda)]\}, \quad (1)$$

where $|n_{\lambda'}\rangle$ are the eigenstates of energy $E_n[\lambda']$ of the final Hamiltonian, while $|0_\lambda\rangle$ is the ground state of $H[\lambda]$. An equivalent representation is obtained in terms of the characteristic function $G(u)=\int e^{-iWu}P(W)dW$ of the distribution $P(W)$, which can be easily found to be related to the Loschmidt echo $\mathcal{L}(u)$ [8,13],

$$G(u) = \langle 0_\lambda|e^{iH[\lambda]u}e^{-iH[\lambda']u}|0_\lambda\rangle = [\mathcal{L}(u)]^*. \quad (2)$$

The Loschmidt echo has emerged previously in the study of x-ray singularities in metals [14], dephasing [15], quantum chaotic behavior [16], and upon a Wick rotation is analogous to a partition function [8]. In addition, its direct computation for a prototype quantum critical system, the Quantum Ising chain, revealed that for global quantum quenches of the transverse field, the presence of criticality leads to singularities of the moments of $P(W)$ as a function of the quench parameters, while for local quenches, $P(W)$ itself displays an edge singularity at low energies [8].

The goal of the present paper is to move one step forward toward elucidating and eventually establishing the connection between the qualitative features of the statistics of the work $P(W)$ and the generic characteristics of a physical system (e.g., its integrability, the presence of a critical point in parameter space). In order to do so, it is important to obtain benchmark results for $P(W)$ in exactly solvable models, understand their main qualitative features, and describe their physical origins [8,17]. With this motivation, we study the statistics of the work done in quantum quenches in the Dicke model [18,19], an exactly solvable Hamiltonian describing an ensemble of two level systems ("atoms") interacting with a single bosonic mode ("photons"). Our motivation to select this model is mainly its simplicity and integrability. However, it is important to mention that the Dicke Hamiltonian was originally proposed to describe the coherent spontaneous emission of radiation in atomic gases within the dipole approximation [20]. Most importantly, in the thermodynamic limit, the Dicke model displays a quantum phase transition [21] at a critical value of the atom-photon coupling separating a normal phase characterized by a vanishing number of excited atoms/photons and a superradiant phase in which the number of excitations scale extensively with the number of atoms [19,20]. The Dicke model describes the physics of a range of physical systems, from molecular magnets [22], to Rydberg atoms coupled to cavity radiation [23]. Despite the fact that the phase transition predicted originally by Hepp and Lieb [18] has not yet been observed experimentally, a number of possible realizations of the Dicke model with critical or close to critical coupling in cavity QED [24] and circuit QED [25] have been recently proposed.

In the following, we consider quenches of one of the available knobs in experimental realization of the Dicke model: the coupling constant. Focusing on quenches within the normal phase, we show that criticality leaves clear signatures on the dynamics of the system and on $P(W)$: as the initial coupling tends to the critical point the average work done on the system tends to diverge, while for quenches ending at criticality the probability distribution displays an interesting sequence of edge singularities. We will give a simple physical picture explaining this difference, ultimately related to the difference between the processes of compression and expansion of a particle subject to a sudden change in its confining potential. We further elucidate these findings by computing exactly the time-dependent statistics of observables such as the quadrature operators of the photon field and the occupation of the bosonic modes. The rest of the paper is organized as follows: we present the model, establish the notations, and discuss the statistics of the work $P(W)$ in Sec. II. Next, we study the statistics of the observables starting with the quadratures of the cavity field and followed by the total occupation of the bosons in Sec. III. Finally, we give our conclusions in Sec. IV.

II. HAMILTONIAN AND THE STATISTICS OF THE WORK DONE

The Dicke model [18] describes the coupling of N two level systems, such as two level atoms, to a single bosonic mode. Its Hamiltonian is

$$H = \sum_{i=1}^N \omega_0\sigma_i^z + \omega a^\dagger a + \frac{\lambda}{\sqrt{N}}\sum_{i=1}^N (a^\dagger + a)(\sigma_i^+ + \sigma_i^-), \quad (3)$$

where the Pauli matrices σ_i describe the dynamics of the two level systems with energy splitting $\hbar\omega_0$, and $a(a^\dagger)$ destroys (creates) a photon of frequency ω (we set $\hbar=1$). The coupling between atoms and photons has been rescaled by $1/\sqrt{N}$ in order to have a well-defined thermodynamic limit $N\to+\infty$. The Dicke model has two phases [18,19], a normal and a superradiant one that are separated by a quantum critical point at $\lambda_c=\sqrt{\omega_0\omega}/2$. The transition as λ exceeds λ_c is characterized by the breaking of parity symmetry leading to the spontaneous generation of an extensive density of photons in the system $\langle a^\dagger a\rangle \propto N$.

The Hamiltonian (3) can be easily diagonalized exactly [18]. Focusing on the normal phase ($\lambda<\lambda_c$) in the thermodynamic limit [19], it is, first of all, convenient to regroup the Pauli matrices into collective spin operators $J^q=\Sigma_{i=1}^N\sigma_i^q$, where $q=z,\pm$. Using the Holstein-Primakoff representation in terms of a bosonic mode b, $J^z=b^\dagger b-N/2$, $J^+=b^\dagger\sqrt{N-b^\dagger b}$, and $J^-=\sqrt{N-b^\dagger b}b$, the semiclassical/thermodynamic limit $N\to+\infty$ can be taken. One obtains

$$H = \omega_0(a^\dagger a + b^\dagger b) + \lambda(a^\dagger + a)(b^\dagger + b) - \frac{N\omega_0}{2}. \quad (4)$$

At this point, the diagonalization proceeds by a standard Bogoliubov rotation as outlined in Ref. [19]. The final form of the Hamiltonian is

$$H[\lambda] = \omega_+ c_+^\dagger c_+ + \omega_- c_-^\dagger c_- + C. \quad (5)$$

Here, the eigenenergies $\omega_\pm$ are given by

$$\omega_\pm(\lambda) = \omega_0\sqrt{1 \pm \frac{2\lambda}{\omega_0}}, \quad (6)$$

while the eigenmodes $c_\pm$ can be expressed in terms of a and b as

$$c_{\pm} = \cosh(r_{\pm})\frac{a \pm b}{\sqrt{2}} + \sinh(r_{\pm})\frac{a^{\dagger} \pm b^{\dagger}}{\sqrt{2}}, \tag{7}$$

with $\tanh(r_{\pm})=(\omega_{\pm}-\omega_0)/(\omega_{\pm}+\omega_0)$. Finally, the constant is $C=[\omega_{+}+\omega_{-}-\omega_0(N+2)]/2$. The vanishing of ω_{-} at $\lambda_c=\omega_0/2$ is a direct consequence of the presence of a quantum critical point. Though we will not consider the superradiant phase at length here, we note that for $\lambda>\lambda_c$, a consistent thermodynamic limit can be taken only after the bosonic operators a,b are displaced [19].

Once the condition of resonance between atoms and photons is imposed, the only dimensionless parameter determining the physics is λ/ω_0, which can be practically varied by either quenching λ or the frequency ω_0. Without loss of generality, we will consider a quantum quench in which the coupling constant is changed abruptly from an initial value λ to a final one λ' (the discussions of quenches of ω_0 is analogous). We will focus on the case where both $\lambda,\lambda'<\lambda_c$. From the point of view of the statistics of the work done, quantum quenches from $\lambda<\lambda_c$ toward $\lambda'>\lambda_c$ (or vice versa) are not very interesting because the generation of a photon density $\propto N$ requires a work that scales extensively with the number of atoms, while fluctuations are expected to scale like $1/\sqrt{N}$, i.e., to be highly suppressed in the thermodynamic limit. On the other hand, we do not expect major changes in the main qualitative results of this paper for quenches with both $\lambda,\lambda'>\lambda_c$: the choice of focusing on $\lambda,\lambda'<\lambda_c$ has the only purpose of allowing us to obtain closed analytic results for the statistics of the work and of other observables.

Let us start our analysis by computing and characterizing qualitatively the statistics of the work done in a quantum quench of the coupling constant from λ to $\lambda'(\lambda,\lambda'<\lambda_c)$. As stated earlier in the introduction, in order to study the probability distribution $P(W)$ of the work W done in a quantum quench, it is convenient to compute its characteristic function $G(u)=\int e^{-iWu}P(W)dW$, given by Eq. (2) where $|0_{\lambda}\rangle$ is now the vacuum of the operators $c_{\pm}$. For $\lambda,\lambda'<\lambda_c$, the operators diagonalizing the final Hamiltonian $\bar{c}_{\pm}$ are related to the initial eigenmodes $c_{\pm}$ by a Bogoliubov rotation

$$\bar{c}_{\pm} = \cosh(\xi_{\pm})c_{\pm} + \sinh(\xi_{\pm})c_{\pm}^{\dagger}, \tag{8}$$

with $\tanh(\xi_{\pm})=[\omega_{\pm}(\lambda')-\omega_{\pm}(\lambda)]/[\omega_{\pm}(\lambda')+\omega_{\pm}(\lambda)]$.

From these definitions, it is evident that we have to compute

$$G(u) = e^{-i\delta E u}\langle 0_{\lambda}|e^{-i[\omega_{+}(\lambda')\bar{c}_{+}^{\dagger}\bar{c}_{+}+\omega_{-}(\lambda')\bar{c}_{-}^{\dagger}\bar{c}_{-}]u}|0_{\lambda}\rangle, \tag{9}$$

where δE is the difference in the ground state energies of the initial and final Hamiltonians. In order to do so, one has first to express $|0_{\lambda}\rangle$ in terms of the vacuum $|0_{\lambda'}\rangle$ of the final eigenmodes $\bar{c}_{\pm}$. Using the definition $c_{\pm}|0_{\lambda}\rangle=0$ together with the Bogoliubov rotation Eq. (8), one obtains the equation $\cosh(\xi_{\pm})\bar{c}_{\pm}|0_{\lambda}\rangle=\sinh(\xi_{\pm})\bar{c}_{\pm}^{\dagger}|0_{\lambda}\rangle$, which implies that

$$|0_{\lambda}\rangle = S_{+}[\xi_{+}]S_{-}[\xi_{-}]|0_{\lambda'}\rangle, \tag{10}$$

where

$$S_{\pm}[z] = e^{-1/2[z c_{\pm}^{\dagger}(\lambda')^2 - z^{*}c_{\pm}(\lambda')^2]}, \tag{11}$$

are single-mode squeezing operators [26]. The state $|0_{\lambda}\rangle$ is, therefore, a squeezed vacuum of the modes $\bar{c}_{\pm}$. In this representation, the Loschmidt echo takes the simple form

$$G(u) = \langle 0_{\lambda'}|S_{+}^{\dagger}[\xi_{+}]S_{-}^{\dagger}[\xi_{-}]S_{+}[\xi_{+}(u)]S_{-}[\xi_{-}(u)]|0_{\lambda'}\rangle, \tag{12}$$

where $\xi_{\pm}(u)=\xi_{\pm}e^{-2i\omega_{\pm}(\lambda')u}$. Using standard formulas for the overlap of squeezed states, we then obtain

$$G(u) = e^{-i\delta E u}G_{+}(u)G_{-}(u), \tag{13}$$

where

$$G_{\pm}(u) = [1 + \bar{n}_{\pm} - e^{-2i\omega_{\pm}(\lambda')u}\bar{n}_{\pm}]^{-1/2}. \tag{14}$$

Here, we introduced the parameters

$$\bar{n}_{\pm} = \sinh^2(\xi_{\pm}) = \frac{[\omega_{\pm}(\lambda') - \omega_{\pm}(\lambda)]^2}{4\omega_{\pm}(\lambda')\omega_{\pm}(\lambda)}, \tag{15}$$

which physically represent the average occupation of the final eigenmodes $\bar{c}_{\pm}$ in the initial ground state $|0_{\lambda}\rangle$.

From these equations, one can immediately deduce that the distribution $P(W)$ has the form

$$P(W) = \sum_{k,l=0}^{+\infty} \mathcal{P}_{+}(2k)\mathcal{P}_{-}(2l) \times \delta[W - \delta E - 2k\omega_{+}(\lambda') - 2l\omega_{-}(\lambda')], \tag{16}$$

where

$$\mathcal{P}_{\pm}(2k) = \frac{1}{\sqrt{1+\bar{n}_{\pm}}}\binom{k-\frac{1}{2}}{k}\left[\frac{\bar{n}_{\pm}}{1+\bar{n}_{\pm}}\right]^{k}. \tag{17}$$

Qualitatively, the distribution $P(W)$ consists of a series of principal peaks separated by $2\omega_{+}(\lambda')$, each followed by a tail of subpeaks separated by $2\omega_{-}(\lambda')$ describing excited $-$ modes (see Fig. 1).

The partial amplitudes $\mathcal{P}_{\pm}$ control the weight of each peak in $P(W)$. The presence of a quantum critical point and its effect on $P(W)$ can be elucidated by studying the asymptotic behavior of $\mathcal{P}_{\pm}(k)$ for large k. Using Stirling's formula $z! \approx \sqrt{2\pi}z^{z+1/2}e^{-z}$ we obtain

$$\binom{k-\frac{1}{2}}{k} = \frac{(2k)!}{2^{2k}(k!)^2} \approx \frac{1}{\sqrt{\pi k}}, \tag{18}$$

from which, for $k \gg 1$, one gets

$$\mathcal{P}_{\pm}(2k) \approx \frac{1}{\sqrt{1+\bar{n}_{\pm}}}\frac{e^{-k/\zeta_{\pm}}}{\sqrt{\pi k}}, \tag{19}$$

$$\zeta_{\pm}^{-1} = \log\left[1 + \frac{1}{\bar{n}_{\pm}}\right]. \tag{20}$$

The scale $\zeta_{\pm}$ controls the decay of the corresponding amplitude. Notice now that the vanishing of $\omega_{-}(\lambda)[\omega_{-}(\lambda')]$ at the critical coupling implies the divergence of $\bar{n}_{-}$ when $\lambda\to\lambda_c$ $(\lambda'\to\lambda_c)$. Therefore, when $\lambda\to\lambda_c$, we have

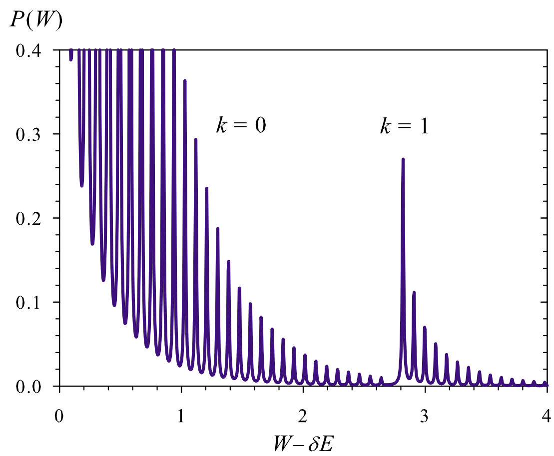

FIG. 1. (Color online) A typical plot of the probability distribution of the work $P(W)$ for a quench from $\lambda=0$ to $\lambda'=0.499\omega_0$. The delta function peaks have been Lorentz-broadened for clarity and work is measured in units of ω_0. Two principal peaks at $W-\delta E=0$, corresponding to $k=0$ in Eq. (16), and $W-\delta E=2\omega_+(\lambda')=2.82\omega_0$, corresponding to $k=1$, are clearly visible. Each principal peak is then followed by a tail of secondary peaks separated by $2\omega_-(\lambda')=0.09$. When $\lambda'=\lambda_c$, the distance between these subpeaks vanishes, leading to their merging which gives rise to an edge singularity at each principal peak (see Fig. 2).

$$\zeta_- \approx \bar{n}_- \approx \frac{\omega_-(\lambda')}{4\omega_-(\lambda)} \propto \sqrt{\frac{\lambda_c}{\lambda_c-\lambda}}, \tag{21}$$

and a similar equation with $\lambda' \leftrightarrow \lambda$ for $\lambda' \to \lambda_c$. The presence of a quantum phase transition in parameter space is marked by the divergence of the scale controlling the exponential decay of the partial amplitude associated with the − modes, which are the ones becoming critical at the transition.

Despite the fact that for both quantum quenches toward the quantum critical point and away from the quantum critical point ζ_- diverges, the physics behind these two situations is deeply different. While the divergence of ζ_- when either λ or λ' is exactly at the quantum critical point always results in a power law decay with power $\alpha=-1/2$ of the partial amplitude $\mathcal{P}_-$ [cf. Equation (19)], this has different effects on the physics of $P(W)$ depending on the type of quench considered. Indeed, when $\lambda \to \lambda_c$, the spacing $\omega_-(\lambda')$ between the secondary peaks associated to each principal peak remains finite. As a result, the slow decay of their amplitudes leads to the divergence (in the thermodynamic limit) of the average work (and of higher moments). On the other hand, the limit $\lambda' \to \lambda_c$ is more subtle: in this case, the spacing $\omega_-(\lambda')$ vanishes and the secondary peaks merge to give rise to power law edge singularities at each principal peak (see Fig. 2). Notice, however, that in this case, the decay of each tail at high energies is exponential, implying that the moments of the distribution remain finite [see Eq. (30) below].

These two cases can be efficiently distinguished by considering the limiting behavior of the energy scale $\Omega_-=2\zeta_-\omega_-(\lambda')$ controlling the decay of the tails of $P(W)$. When the quench is toward criticality ($\lambda' \to \lambda_c$), this scale remains finite

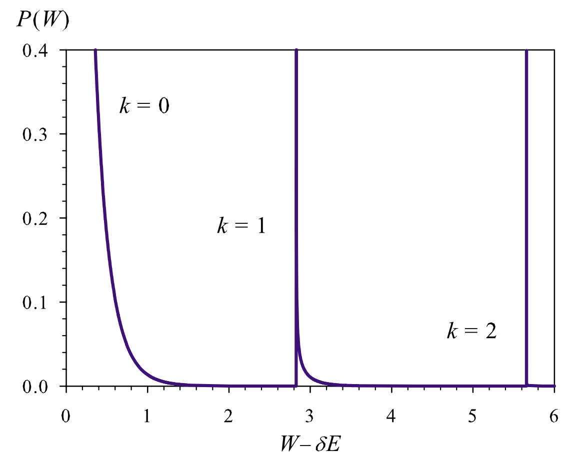

FIG. 2. (Color online) The probability distribution of the work $P(W)$ for a quench from $\lambda=0.1\omega_0$ to $\lambda'=\lambda_c=\omega_0/2$. Here, work is measured in units of ω_0. A sequence of edge singularities described by Eqs. (30) and (31) originating from the merging of the satellite peaks at $\omega_-(\lambda')$ and located at $W-\delta E=\sqrt{8}\omega_0 k$, with integer k, signals the criticality of the system in the final state.

$$\Omega_- \simeq \frac{\omega_-(\lambda)}{4}. \tag{22}$$

On the other hand, for quenches starting infinitesimally close to criticality ($\lambda \to \lambda_c$), it diverges

$$\Omega_- \simeq \frac{[\omega_-(\lambda')]^2}{4\omega_-(\lambda)} \propto \frac{\lambda_c}{\lambda_c-\lambda}. \tag{23}$$

Let us now explore in more detail the difference between quenches toward criticality and away from it giving a simple physical picture to explain their physics. For quenches starting at criticality ($\lambda \to \lambda_c$), the divergence of Ω_- implies the divergence of the average work done on the system. Physically, this can be understood resorting to the coordinate representation of the Hamiltonian representing the − mode,

$$H_-[\lambda] = \omega_-(\lambda)c_-^\dagger c_- \simeq \frac{p_-^2}{2} + \omega_-(\lambda')^2\frac{x_-^2}{2} + \text{const}. \tag{24}$$

In this representation we see that when a quench starts exactly at criticality the − mode corresponds to a completely delocalized free particle. A quench to $\lambda' \neq \lambda_c$ can be understood as the switching on of an harmonic potential tending to confine the mode in a finite volume. This process is conceptually similar to the compression of a gas occupying initially an infinite volume into a finite one: on this basis, we expect the average work done to diverge as $\lambda \to \lambda_c$. This can be readily obtained from our formulas using the characteristic function $G(t)$ to extract directly the cumulants K_n of the distribution $P(W)$ using the standard cumulant expansion $G(u)=\exp[\Sigma_{n=1}^{\infty}(-iu)^n/n!K_n]$. Expanding Eq. (14) to first order, we obtain

$$\langle W \rangle = K_1 = \delta E + \omega_+(\lambda')\bar{n}_+ + \omega_-(\lambda')\bar{n}_-. \tag{25}$$

The average excess work $\langle \delta W \rangle = \langle W \rangle - \delta E$ is then

$$\langle \delta W\rangle = \frac{[\omega_+(\lambda') - \omega_+(\lambda)]^2}{4\omega_+(\lambda)} + \frac{[\omega_-(\lambda') - \omega_-(\lambda)]^2}{4\omega_-(\lambda)}. \quad (26)$$

From this expression, it is evident that quantum quenches are not reversible processes, $W(\lambda\to\lambda') \neq -W(\lambda'\to\lambda)$, and that for $\lambda\to\lambda_c$, one has

$$\langle \delta W\rangle \sim \frac{[\omega_-(\lambda')]^2}{\sqrt{2\omega_0}}\frac{1}{\sqrt{\lambda-\lambda_c}}, \quad (27)$$

which diverges at criticality as anticipated. A similar divergence is observed in the second moment of $P(W)$, which can be easily computed to find

$$\langle(\delta W)^2\rangle = 2[\omega_+(\lambda')]^2\bar{n}_+(1+\bar{n}_+) + 2[\omega_-(\lambda')]^2\bar{n}_-(1+\bar{n}_-). \quad (28)$$

A diametrically opposite situation is obtained for quenches starting with $\lambda\neq\lambda_c$ and ending exactly at criticality. In this case, the corresponding confining harmonic potential in the coordinate representation of the initial Hamiltonian (24) associated with the mode $\omega_-(\lambda)\neq 0$ is removed after the quantum quench. This process is conceptually similar to the free expansion of a gas in vacuum and is, therefore, not expected to be characterized by a divergence of the average work. However, it turns out that the resulting distribution $P(W)$ displays a series of edge singularities resulting from the fact that Ω_- stays finite even in the limit $\lambda'\to\lambda_c$ [see Eq. (22)]. The most elegant way to obtain this result is by observing that for $u \ll 1/\omega_-(\lambda')$, we can always approximate

$$G_-(u) \simeq (1 + i\Omega_- u)^{-1/2}, \quad (29)$$

whose Fourier transform is

$$P_-(w) = \frac{\Theta(w)}{\sqrt{\pi w \Omega_-}}e^{-w/\Omega_-}. \quad (30)$$

Using these expressions for $\lambda'=\lambda_c$, one immediately obtains

$$P(W) = \sum_{k=0}^{+\infty} \mathcal{P}_+(2k)P_-(W - \delta E - \sqrt{8}\omega_0 k), \quad (31)$$

which has the expected form (see Fig. 2). It is interesting to notice that edge singularities in the statistics of the work have been previously reported in local quenches of the transverse field in a Quantum Ising chain [8]. These two examples seem to suggest a connection between criticality and edge singularities in the statistics of the work done in quantum quenches, a topic that deserves a deeper study in the future.

III. STATISTICS OF OTHER OBSERVABLES

Let us now continue the characterization of quantum quenches in the Dicke model by focusing on the statistics of observables such as the quadrature operators and the occupation of the bosonic modes. In contrast with the statistics of the work, which is time-independent, the statistics of generic observables depends on the time t elapsed after the quench. More explicitly, let us consider a generic observable $\hat{Q}$ having eigenstates $|n\rangle$ with corresponding eigenvalues q_n. If the initial state before the quench is $|\Phi_0\rangle$, the probability to obtain q as a result of the measurement of $\hat{Q}$ at time t is

$$P(q,t) = \sum_n |\langle n|e^{-iH_f t}|\Phi_0\rangle|^2\delta(q-q_n), \quad (32)$$

where H_f is the final Hamiltonian. Hence, the characteristic function $G_Q(u,t)=\int e^{-iqu}P(q,t)dq$ of the distribution $P(q,t)$ is given by the expression

$$G_Q(u,t) = \langle\Phi_0|e^{-i\hat{Q}(t)u}|\Phi_0\rangle, \quad (33)$$

where $\hat{Q}(t)=e^{iH_f t}\hat{Q}e^{-iH_f t}$.

Let us now compute explicitly this characteristic function for two important observables for the Dicke model: the quadratures given by

$$X(\alpha) = \frac{1}{\sqrt{2}}(ae^{-i\alpha} + a^\dagger e^{i\alpha}), \quad (34)$$

and the total occupation of the bosonic modes

$$N = a^\dagger a + b^\dagger b, \quad (35)$$

which can be interpreted as the order parameter of the transition.

A. Quadrature operators

Let us start by computing the statistics of the quadrature operator $X(\alpha)$ for a generic quench starting at λ and ending at λ' with $\lambda,\lambda'<\lambda_c$. The appropriate characteristic function is

$$G_\alpha(u,t) = \langle 0_\lambda|e^{iH[\lambda']t}e^{-iX(\alpha)u}e^{-iH[\lambda']t}|0_\lambda\rangle. \quad (36)$$

Since the final Hamiltonian is diagonalized by the modes $\bar{c}_\pm$, it is convenient to first express the operator $X(\alpha)$ in Eq. (34) in terms of them. Inverting a transformation analogous to Eq. (7) (with $\lambda\to\lambda'$) yields

$$a = \frac{1}{\sqrt{2}}[\cosh(\bar{r}_+)\bar{c}_+ + \cosh(\bar{r}_-)\bar{c}_- - \sinh(\bar{r}_+)\bar{c}_+^\dagger - \sinh(\bar{r}_-)\bar{c}_-^\dagger], \quad (37)$$

where the bars mean that the corresponding quantities are to be evaluated using the final value of the coupling parameter λ'. Since the $\bar{c}_\pm$ operators evolve trivially in time according to the final Hamiltonian, we find that the characteristic function has the simple expression

$$G_\alpha(u,t) = \langle 0_\lambda|e^{-iX_+(\alpha,t)u}e^{-iX_-(\alpha,t)u}|0_\lambda\rangle, \quad (38)$$

where

$$X_\pm(\alpha,t) = \frac{A_\pm(\alpha)}{2}e^{-i\omega_\pm(\lambda')t}\bar{c}_\pm + \frac{A_\pm^*(\alpha)}{2}e^{i\omega_\pm(\lambda')t}\bar{c}_\pm^\dagger, \quad (39)$$

with

$$A_\pm(\alpha) = [e^{-i\alpha}\cosh(\bar{r}_\pm) - e^{i\alpha}\sinh(\bar{r}_\pm)]. \quad (40)$$

The next step in obtaining a closed form result consists of expressing the state $|0_\lambda\rangle$ in terms of the vacuum of the operators $\bar{c}_{\pm}$. Using the result of the previous section [Eq. (10)], we obtain

$$G_\alpha(u,t) = \langle 0_{\lambda'} | e^{-i\tilde{X}_+(\alpha,t)u} e^{-i\tilde{X}_-(\alpha,t)u} | 0_{\lambda'} \rangle, \quad (41)$$

where $\tilde{X}(\alpha,t)=S^\dagger(\xi_\pm)X_\pm(\alpha,t)S(\xi_\pm)$. Finally, using the formula $S^\dagger(\xi)cS(\xi)=\cosh(\xi)c+\sinh(\xi)c^\dagger$, we obtain

$$G_\alpha(u,t) = \langle e^{\eta_+ u\bar{c}_+ - \eta_+^* u\bar{c}_+^\dagger} e^{\eta_- u\bar{c}_- - \eta_-^* u\bar{c}_-^\dagger} \rangle, \quad (42)$$

with

$$\eta_\pm(\alpha) = -\frac{i}{2}[A_\pm(\alpha)e^{-i\omega_\pm t}\sinh(\xi_\pm) + A_\pm^*(\alpha)e^{i\omega_\pm t}\cosh(\xi_\pm)]. \quad (43)$$

Since the exponentials appearing in the expression Eq. (13) for the characteristic function are standard displacement operators, taking their vacuum expectation value gives [26]

$$G_\alpha(u,t) = e^{-u^2/2[|\eta_+(\alpha,t)|^2 + |\eta_-(\alpha,t)|^2]}. \quad (44)$$

The statistics of the quadrature operators maintains the characteristics it has in the initial state and is always Gaussian. The only scale characterizing the distribution is its variance, which is given by

$$\langle \delta X(\alpha)^2 \rangle = |\eta_+(\alpha,t)|^2 + |\eta_-(\alpha,t)|^2. \quad (45)$$

Using this expression, we may get further insight on the difference between quenches that are driven toward criticality and those that start near criticality and then are driven away from it. Indeed, focusing on the case $\alpha=0$ that corresponds to the "coordinate" operator $X(\alpha=0)$, we have

$$|\eta_\pm(0,t)|^2 = \omega_0 \left\{ \frac{\cos[\omega_\pm(\lambda')t]^2}{4\omega_\pm(\lambda)} + \frac{\omega_\pm(\lambda)\sin[\omega_\pm(\lambda')t]^2}{4[\omega_\pm(\lambda')]^2} \right\}. \quad (46)$$

Notice now that in the case in which the initial state is close to critical, the closer λ is to λ_c, the more delocalized the mode $\bar{c}_-$ is, initially. This results in the divergence of the amplitude of the oscillations of $\langle \delta X(\alpha)^2 \rangle$ as $1/\sqrt{\lambda_c - \lambda}$ when $\lambda \to \lambda_c$. On the other hand, when the final coupling constant λ' approaches criticality, one is describing the physics of an initially confined mode $\bar{c}_-$ that is "released" at $t=0$: it is, therefore, not surprising that for large times t the width of the distribution increases linearly with time when $\lambda'=\lambda_c$, that is,

$$\sqrt{\langle \delta X(\alpha)^2 \rangle} \simeq \frac{\sqrt{\omega_0 \omega_-(\lambda)}}{2} t. \quad (47)$$

B. Occupation number

The statistics of the total occupation of the bosonic modes

$$N = a^\dagger a + b^\dagger b, \quad (48)$$

turns out to encode similar information. Let us compute the associated characteristic function

$$G_N(u) = \langle 0_\lambda | e^{iH[\lambda']t} e^{-iNu} e^{-iH[\lambda']t} | 0_\lambda \rangle. \quad (49)$$

First of all, we express the operator N in terms of the modes diagonalizing $H[\lambda']$. Using Eq. (7), we obtain $N=N_+ + N_-$, where

$$N_\pm = \cosh^2(\bar{r}_\pm)\bar{c}_\pm^\dagger \bar{c}_\pm + \sinh^2(\bar{r}_\pm)\bar{c}_\pm \bar{c}_\pm^\dagger - \sinh(\bar{r}_\pm)\cosh(\bar{r}_\pm) \times \{ (\bar{c}_\pm^\dagger)^2 + (\bar{c}_\pm)^2]\}. \quad (50)$$

Evolving this operator in time and expressing the initial state $|0_\lambda\rangle$ in terms of the vacuum $|0_{\lambda'}\rangle$ of the operators $\bar{c}_\pm$ as in Eq. (10), we obtain $G_N(u)=G_{N_+}(u)G_{N_-}(u)$, where

$$G_{N_\pm}(u) = e^{iu/2} \left\langle \exp\left[\sum_{j=1}^{3} \gamma_j(\pm) K_j(\pm) \right] \right\rangle. \quad (51)$$

Here,

$$K_1(\pm) = K_2^\dagger(\pm) = \frac{(\bar{c}_\pm^\dagger)^2}{2}, \quad (52)$$

$$K_3(\pm) = \frac{\bar{c}_\pm^\dagger \bar{c}_\pm + \bar{c}_\pm \bar{c}_\pm^\dagger}{2}, \quad (53)$$

and

$$\gamma_1(\pm) = \gamma_2(\pm)^* = -iu\{\cosh(2\bar{r}_\pm)\sinh(2\xi_\pm) - \sinh(2\bar{r}_\pm)\cosh(2\xi_\pm)\cos[2\omega_\pm(\lambda')t] - \sinh(2\bar{r}_\pm)\sin[2\omega_\pm(\lambda')t]\}, \quad (54)$$

$$\gamma_3(\pm) = -2iu\{\cosh(2\bar{r}_\pm)\cosh(2\xi_\pm) - \sinh(2\bar{r}_\pm)\sinh(2\xi_\pm)\cos[2\omega_\pm(\lambda')t]\}. \quad (55)$$

In order to compute the matrix elements in Eq. (51), we notice that for both + and − modes, the operators K_j form a closed algebra with commutation relations $[K_1,K_2]=-2K_3$, $[K_1,K_3]=-K_1$, and $[K_2,K_3]=K_2$. We may then apply a standard operator ordering theorem [26] stating that for this algebra of operators, the equality

$$e^{\sum_{j=1}^{3}\gamma_j K_j} = e^{\Gamma_1 K_1} e^{\ln(\Gamma_3)K_3} e^{\Gamma_2 K_2}, \quad (56)$$

holds, where

$$\Gamma_{1,2} = \frac{2\gamma_{1,2}\sinh(\beta)}{2\beta\cosh(\beta) - \gamma_3\sinh(\beta)}, \quad (57)$$

$$\Gamma_3 = \frac{1}{\left[\cosh(\beta) - \frac{\gamma_3}{2\beta}\sinh(\beta)\right]^2}, \quad (58)$$

with $\beta^2=\gamma_3^2/3-\gamma_1\gamma_2$. Using Eq. (56) together with Eq. (52), we easily obtain

$$\left\langle \exp\left[\sum_{j=1}^{3} \gamma_j(\pm) K_j(\pm) \right] \right\rangle = [\Gamma_3(\pm)]^{1/4}. \quad (59)$$

Some straightforward algebra now shows that in the present case $\beta^2=-u^2$, and hence a direct computation of Γ_3 leads us to

$$G_{N_{\pm}}(u) = \frac{e^{iu/2}}{\sqrt{\cos(u) + ig_{\pm}(t)\sin(u)}}, \tag{60}$$

where

$$g_{\pm}(t) = \frac{1}{2}\left[\frac{\omega_{\pm}(\lambda')^2}{\omega_0\omega_{\pm}(\lambda)} + \frac{\omega_0\omega_{\pm}(\lambda_0)}{\omega_{\pm}(\lambda')^2}\right]\sin^2[\omega_{\pm}(\lambda')t] + \frac{1}{2}\left[\frac{\omega_{\pm}(\lambda)}{\omega_0} + \frac{\omega_0}{\omega_{\pm}(\lambda)}\right]\cos^2[\omega_{\pm}(\lambda')t]. \tag{61}$$

The only parameters entering the characteristic function are $g_{\pm}(t)(g_{\pm}\geq 1)$. Physically, they characterize the average occupation of the bosonic modes. Indeed, taking the first logarithmic derivative of the characteristic function leads to

$$\langle N\rangle = \frac{g_+(t) + g_-(t)}{2} - 1. \tag{62}$$

As for the average work and the quadrature variance $\langle \delta X(\alpha)^2\rangle$, the behavior of the occupation for quenches starting close to criticality and going toward criticality is deeply different. When $\lambda' \to \lambda_c$, we indeed have that for large times

$$\langle N\rangle \simeq \frac{\omega_0\omega_-(\lambda)}{2}t^2, \tag{63}$$

while in the second case $(\lambda \to \lambda_c)$, the amplitude of the oscillations diverges as $\sqrt{\lambda_c/(\lambda_c-\lambda)}$.

Finally, we give the result for the full distribution of the occupation number N. By taking the Fourier transform of Eq. (60), one may easily obtain

$$P(N) = \sum_{M=0}^{+\infty} \mathcal{N}(M)\delta(N-2M), \tag{64}$$

where the weights $\mathcal{N}(M)$ are given by the finite sums

$$\mathcal{N}(M) = \sqrt{\frac{4}{(1+g_+)(1+g_-)}}\sum_{k=0}^{M}\binom{k-\frac{1}{2}}{k}\binom{M-k-\frac{1}{2}}{M-k} \times \left(\frac{g_+-1}{g_++1}\right)^{2k}\left(\frac{g_{-1}-1}{g_-+1}\right)^{2(M-k)}. \tag{65}$$

IV. CONCLUSIONS

In this paper, we studied the statistics of the work and other observables for quantum quenches of a prototypical quantum critical system, the Dicke model. Focusing on quenches of the coupling constant from an initial value λ to a final one λ' in the normal phase $(\lambda,\lambda' < \lambda_c)$, we computed exactly the characteristic function of the probability distribution of the work, as well as that associated with the statistics of the quadratures of the cavity modes and with the total occupation of the bosonic modes. We found that while criticality always leaves an imprint on the statistics of observables, there is a deep difference between quenches starting close to the critical point and those ending close to it. In the first case, the average work (as well as the amplitude of the oscillations of the variance of the quadratures and of the average occupation) diverges as criticality is approached. In contrast, for quenches toward the quantum critical point the moments of the distribution of the work stay finite, while the distribution itself displays a sequence of edge singularities. This occurrence is accompanied by a characteristic quadratic growth in time of the variance of the "coordinate" operator associated with the cavity field, and a similar quadratic temporal growth of the average number of bosonic modes in the system. We developed a simple physical picture explaining the origin of these effects: for quenches starting close to criticality, the divergences are caused by a mode initially delocalized in phase space subject to a final confining potential that tends to compress it in a finite volume. The situation is opposite for quenches toward criticality: the initially localized mode is released at $t=0$ and is allowed to spread coherently in phase space. On the basis of this general qualitative picture, we expect our main qualitative findings listed above to apply also for $\lambda,\lambda' > \lambda_c$.

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